# What is <br> Calculus? 



A BEDTIME STORY

## Kamex, <br> You opened my eyes, changed my world, and fixed my computer. Thank you.

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## Directions

This little book actually contains a lot of complex ideas. If you are reading it because you will soon be taking a calculus course you should read it in short sections before bedtime, so those ideas can sink in while you sleep. If you have no intention of taking a calculus course you should also read this book in short sections before bedtime, because mathematics is the greatest natural cure for insomnia.

## Once Upon a Time

Once upon a time Tom drove a distance of 130 miles at a constant speed, and his trip took 2 hours. What was his speed? Since speed is distance (the change in position) divided by time, Tom's speed was 65 miles per hour. That is a textbook problem, and it might as well be a fairy tale. In real life, Tom would inevitably encounter some type of farm equipment that has to be on the highway for no apparent reason, or get stuck behind a truck that has sped up to 56 miles per hour in an attempt to overtake another truck traveling at 55 miles per hour. As Tom hits his brake (muttering something we won't include here), or when he speeds up to make up for lost time, his exact speed at a particular moment would be very difficult to calculate. And without an accurate knowledge of that speed, we cannot predict when he will actually arrive at his destination.

The world around you is always changing. Some changes happen in the blink of an eye, while others are so slow you don't even see them happening, like your nails growing or your favorite shoes wearing out. Almost none of these changes happen at a perfectly constant rate. Even the planets don't move around the sun at a constant speed. Back in the $17^{\text {th }}$ century, the scientific revolution was in full swing as people tried to explain all kinds of phenomena in the natural world. Regular math was just not meeting their needs. The challenge for mathematics was to come up with a way to do calculations involving "changing" change. Today we have that way. If an enormous asteroid is about to hit Earth, you'll be counting on some nerdy-looking person with glasses to use calculus to figure out how that asteroid could be intercepted and thrown off course.

In this story we are going to take a close look at how we can understand change by using calculus. Although calculus has a scary reputation, the basic idea behind it is really not terribly complicated. If we are dealing with non-constant change, an odd shape, or a curve, we can make meaningful calculations by dividing the item in question up into infinitely many infinitely small parts. Because each part is infinitely small, any change that occurs during or within that part itself is negligible. Surprisingly, we can reach perfectly accurate conclusions this way. Over time this relatively simple idea has become layered with complexity, and the bits and pieces of the original calculus are mixed in with newer stuff. Math history is actually less boring than mathematics itself, and in this case a little history can make things a lot easier to understand. We're going to borrow a few original bits from the very first calculus textbook, Analyse des Infinitement Petits (Analysis of the Infinitely Small), written by the Marquis de L'Hospital. If you already know some calculus this approach may look strange to you, but the simplicity of the
original makes it worth spending a little extra time. Also, it will make it much easier to learn integration, the second part of calculus.


In many areas beaches are eroding, and sand is brought in by the truckload from elsewhere in an attempt to preserve nice tourist areas that generate a lot of revenue. The only way to increase the size of a beach is by adding sand, and the smallest amount of sand we can add is a single grain. Now suppose that you are relaxing on the beach, and I come by with a little toy pail and drop a single grain of sand down in front of you. If I then ask you whether I had just increased the size of the beach, what would you say? Aside from telling me that I am crazy, some people might say that it didn't really make a difference to add so little, while others would think that theoretically the beach is now larger. Either way, the question would help you realize that the number of grains of sand on the beach is incredibly large.

An extremely tiny change in a beach could be counted in terms of grains, and an extremely tiny change in the amount of water in the ocean could be counted in terms of drops. In calculus however we will be imagining change as occurring in infinitely small increases or decreases. Take some quantity $x$, which could represent a distance, area, volume, a length of time, etc., and consider an infinitely small change in it. This infinitely tiny change, less than a single grain of sand on a beach, or even a single drop of water in an entire ocean, is called dx . The term dx is an abbreviation for "the difference in $x$ ", now officially called the differential of $x$. The first part of calculus is actually called differential calculus, but because "differential" sounds complicated I will just use the original word "difference" instead. In algebra, when you place two letters next to each other it is assumed that they represent quantities that should be multiplied. That is not the case for dx ; dx is a single entity that is represented by two letters. Anyway, if the quantity $x$ increases by an infinitely tiny amount, its size would be $x+d x$ :


If we had some other quantity $y$, the infinitely small difference of $y$ would be dy. If $y$ increases by an infinitely small amount, its size would be $y+d y$ :

## y



Another way of looking at this is that dy is the difference, or the new size minus the old size: $y+d y-y=d y$. You should keep in mind that the difference could be negative, since $y$ (or $x$, or any other quantity) could be decreasing instead of increasing. In calculus you may express the difference of anything by putting a din front of it. If you have some area $A$, then the difference of $A$ is called $d A$, and it represents an infinitely small change in the area. $d V$ would be the infinitely small difference of a volume V , and so on. Not too hard so far.

Although Isaac Newton is credited with discovering calculus, the " d "-notation that we use today came from Gottfried Leibniz who independently developed calculus a little later.


Isaac Newton


Gottfried Leibniz

In algebra, $x$, $y$ and other letters may simply represent fixed unknown quantities. In that case we can solve for them and find one or two values that will "fit" an equation. Now that you have reached calculus you will use such letters to represent quantities that can increase or decrease. Isaac Newton called these changing, or "flowing", quantities "fluents". Letters near the end of the alphabet will represent changing quantities. Lowercase letters at the beginning of the alphabet like $a, b$, and especially $c$, are often used for constants. Since the definition of a constant is that it never increases or decreases, the difference of a constant is always zero. To illustrate this important point we will look at an example that is, sadly, based on a true story:

When Myra checks out her new college calculus textbook, she notices that it contains 4860 different problems. If the number of problems in Myra's textbook is represented by the letter $P$, find $d P$.

Sometimes people will rip pages out of a magazine so they can keep a good article or a recipe, but it seems unlikely that anyone would steal a page from Myra's calculus book to obtain a particularly exciting problem. We expect P to stay constant, even when we consider all other possible variables. P will be the same now or at the end of Myra's course, if she keeps or sells her book, or if the book ends up being used for a different course at another college. dP should
be the difference, or the "new" value minus the "old" value. $d P=P+d P-P$. Since the new and old values are identical for the constant $P$, regardless of how we define "new" and "old", $d P$ is zero. $d P=d(4860)=0$. Notice that $d P$ is zero even though $P$ is quite a large number.

As we said earlier, an infinitely tiny change in the quantity x is represented by dx . Implied in this is that we can divide $x$ into infinitely many infinitely small parts, all of size $d x$.

Ideally the only difficulty in calculus should be the term "infinitely small". How small is that exactly? Some people say that infinitely small is really zero. I would prefer that too so I don't get dizzy thinking about it. Unfortunately, Kamex ruined that idea for me. To show me that my point of view was wrong, he took two cereal boxes and handed me one. "We will both cut our boxes into infinitely many slices," he announced. He moved his hands to indicate horizontal cuts, all the way from the top of each box to the bottom to produce rectangular slices. "There, now we both have infinitely many slices. My slices are infinitely thin. Your slices have zero thickness, right?" I nodded. "The only difference is that I can put my box back together, and you can't," he said. Then he smiled pleasantly and walked away. As I stared pointlessly at a cereal box that had just vanished in front of my eyes, it occurred to me that even though the difference between infinitely small and zero is literally next to nothing, it is still very important. It also occurred to me that if I was going to argue with Kamex I should be better prepared. Anyway, when we divide a variable quantity like $x$ into infinitely many infinitely small parts, we want to be able to put it back together. That is, the sum of all of those infinitely tiny dx 's must bex.

Just like "infinitely large", infinitely small is a concept rather than a specific size. In calculus the sum of the infinitely small parts is abbreviated as S , but back when calculus was discovered a special elegant letter $S$ was used to represent this sum: $\int$. So, when we go to put $x$ back together we take the $\int u m$ of all the $d x$ 's, which is abbreviated as $\int d x$. Now we can say that $\int d x=x$. This just states that $x$ is the sum of the infinitely many, infinitely small $d x$ 's that we have divided it into. In the same way $\int d y=y$ and $\int d A=A$. In calculus, the sum is called an integral. Just like an integer is a whole number, an integral is the whole thing.

If you increase something by an infinitely tiny amount, is it really larger than it was before?? The odd idea that $\mathrm{x}+\mathrm{dx}$ is equal to x is what makes calculus possible. We can add differences, multiply them, and even divide by them. Then we say that $x+d x=x$, which makes $d x$ just disappear when it is no longer required. Once the idea was there it was surprisingly easy to set up the framework of this new math. Calculus worked, for constant change as well as changing change, and it provided the wings on which Newton's imagination soared through the solar system. Science advanced quickly as people in Britain and Europe began using this powerful new tool to help them solve problems that had been beyond their reach before.

If you are still awake at this point, now may be a good time to head off to bed. Calculus works, but saying that x plus dx equals x although dx isn't really zero does seem rather odd.

After the idea of calculus became widely known Bishop Berkeley mocked differences as "ghosts of departed quantities", and called conclusions based on them invalid: "For when it is said, let the Increments vanish, i.e. let the Increments be nothing, or let there be no Increments, the former supposition that the Increments were something, or that there were Increments, is destroyed, and yet a consequence of that supposition, i.e. an expression got by virtue thereof, is retained." Berkeley is saying that differences are either there or not. You can't just use them to draw some conclusion and then have them conveniently disappear. Because he did have a point there, mathematicians eventually brought in the concept of limits to make the basic idea of calculus work in a more elegant way. Instead of the infinitely small change in $x$ that is called dx , they considered a regular small change in x . "The change in x " was called $\Delta \mathrm{x}$, using the Greek letter $d$, delta. Now imagine that you have $x+\Delta x$, and $\Delta x$ gets smaller and smaller. The limit of this process is that you would eventually just have $x$. A very careful definition of the word "limit" provided a counter-argument to Bishop Berkeley's objection. You will learn a lot about limits in your calculus course. The point of this is to avoid, or at least sidestep, the problem of having to say that $\mathrm{x}+\mathrm{dx}=\mathrm{x}$ so that dx must be nothing.

An alternative and completely unofficial explanation of $x+d x=x$ is provided below for your entertainment. Please read it critically and form your own ideas. In fact, you should probably do that with everything you read.

Because infinity (represented by the symbol $\infty$ ) is a concept rather than a number, it has some very peculiar properties. For example, there are infinitely many counting numbers. However, half of those are odd numbers, and there are also infinitely many of them. The other half are even numbers, and again there are infinitely many of those. It seems that $\infty \div 2=\infty$, and $\infty+\infty=\infty$. If you have infinitely many of something, and you add one more, you still have infinitely many items: $\infty+1=\infty$.

Infinitely small has some of these same peculiar properties, such as "infinitely small" $\div 2=$ "infinitely small". Following the cereal box incident, I spent many months reading and thinking about these properties, and eventually I got the feeling that I was missing something. I would need to speak with Kamex again, but this time I would be ready. As usual he was busy reading a Wikipedia article, probably about some
obscure topic like the evolution of whales or the history of the Korean alphabet. "I would like to talk about infinitely small," I said, without bothering with preliminaries that he wouldn't appreciate anyway. "Why not talk about infinitely large?" he replied. Although the answer was instant, he was still reading. If I didn'tcatch his interest soon there would be no discussion. With no time to make something up, I tried the truth: "Infinitely large scares me because it is sobig." That worked. He actually stood up and walked toward me. The invisible barrier that always seemed to separate him from the rest of the world disappeared briefly as he looked directly at me, his eyes filled with compassion. Afterwards I was never sure if he felt sorry for me because of my irrational fear, or because of the depth of my ignorance. "You do realize," he said gently, "that if you actually had something infinitely small, everything else would become infinitely large by comparison." Then he turned abruptly and wentback to his reading. The conversation I had spent months preparing for was over in less than sixty seconds. It didn't matter. I had what I came for.

Iknew that Kamex was right, and I could even see it on my computer. If you draw a circle on your screen, pick a point on the edge, and start zooming in to see something infinitely small, the circle will get bigger and bigger. By the time you would be looking at an infinitely small part of the circle, the circle itself would be infinitely large. Many years later, when the introduction to Analysedes Infinitement Petits had finally been translated, I discovered that the Marquis de L'Hospital had also been quite impressed by the ability of infinitely small things to create such comparative infinities.

So now should I start worrying about infinitely small monsters hiding under my bed? Well, there probably is no such thing as "infinitely small" in the real world. The universe appears to have a smallest unit of time, called the Planck time (probably about $5.4 \times 10^{-44}$ seconds) and a smallest unit of distance, called the Planck length (thought to be about $1.6 \times 10^{-35}$ meters). Mathematics is not restricted by such realities however, and we can imagine that $d x$ exists as a theoretical infinitely small change in the changing quantity $x$. Now consider $x+d x$. By placing an infinitely small increment in $x$ next to the variable $x$, we have actually made $x$ infinitely large by comparison. Most of the time this goes unnoticed and doesn't cause any problems. A picture of an infinitely large object drawn on paper looks the same as a picture of a regular object, and the paper may be safely disposed of at your local recycling facility. However, if you do multiple calculations involving the same infinitely large object, or compare two such objects, you could run into an inconvenient paradox.
add as it may seem, $x+d x=x$; that is, adding $d x$ to $x$ does not change its size even though dx is not equal to zero. If $x$ has temporarily become infinitely large, that may not really be a contradiction. Another way of looking at this is to say that $x$ is a set (a collection of things) that contains an infinite number of increments of size $d x . x+d x$ is a set that contains an infinite number of increments of size dx, plus one. Both sets are infinitely large, so the additional dx does not contribute anything to the total size of $x$.

When we are dealing with infinity, we tend to toss the concept around in a casual manner. One infinity is just like another infinity - they all look the same to us (even though georg Cantor already proved that some
infinities are much larger than others). Fortunately we are more careful with the infinitely small differences we use in calculus. The key seems to be to establish a relationship between variables before we shrink them down to something infinitely small. For example, we could start by saying that the changing quantity $y$ is twice as large as the changing quantity $x$. Then we name the infinitely small changes in these quantities as $d y$ and $d x$ so that we can compare them and distinguish them from each other, even though they are both "infinitely small".

To better understand change, we make graphs of it. Steady change usually graphs as a straight line, while "changing" change ends up looking like a curve of some sort on our graphs. When we use the idea of an infinitely small change along with such curves, it is effectively like zooming in on the curve:


An infinitely small piece of a curve is just like a straight line.
These are the basic ideas of differential and integral calculus. They have been expanded to help people solve many different kinds of problems.

Now you can look at the world in a whole new way. As things change, you can imagine their differences (differentials). Some things have negative differentials. If the volume of an ice cube is represented by V , what do you imagine dV looks like as the ice cube melts?

## Check Your Skills

1. Suppose that the amount of air in an air mattress is represented by the letter A. What is the difference (differential) of A?
2. The letter $h$ represents the height of a plant. What is $\int d h$ ?
3. If $\mathrm{c}=12$, what is dc ?

Answers are located in the next Check Your Skills section.

## Addition and Subtraction

Once people had the idea that $x+d x=x$, they started to create a system of calculus math. That required addition, subtraction, multiplication, division and a way to work with exponents. Surprisingly, none of this needed more than a good knowledge of arithmetic and some basic algebra. The easiest math to start with is addition.

A line with length x grows by adding infinitely tiny amounts dx to the length:

How does $x+5$ grow?


$$
d z=d x+0
$$

Since the difference of 5 is zero, the only change here is the one that occurs in $\mathrm{x} . \mathrm{dz}=\mathrm{dx}$. If a constant is involved in addition or subtraction, it just disappears when you take the difference. The difference does not show the constant.

Now suppose that $\mathrm{z}=\mathrm{x}+\mathrm{y}$ :

$$
z=x+y
$$

$$
z+d z=x+d x+y+d y
$$

$$
d z=d x+d y
$$

We can draw the same conclusion using algebra. dz is the change, or the new value minus the old value:
$d z=(z+d z)-z=(x+d x+y+d y)-(x+y)=d x+d y$
In the same way, if $z=x-y$ then $d z=d x-d y$. Differences may simply be added and subtracted.

## Check Your Skills

Answers to previous Check Your Skills: 1. dA 2. h, which is the height of the plant, not the plant itself. 3. 0

1. If $y=x-10$, then $d y=$ ?
2. If $z=x+w$, then $d z=$ ?
3. If $z=x+x$, then $d z=$ ?

## Multiplication

Now that we know how to do addition and subtraction with differences, we'll try multiplication. The discoveries described in this section were made using basic algebra. That works, but it's a bit long. It can also make it seem quite mysterious what actually happens with dx and dy . We'll use some pictures instead so we can see what is going on.

If $y=3 x$, we might expect that this relationship would be maintained so that $d y$ is three times as large as dx : $\mathrm{dy}=3 \mathrm{dx}$ (even though both dy and dx are infinitely small). We already saw how x grows, so now let's look at $3 x$ :


As this grows, you can see that the difference is indeed 3dx:


If $y=3 x$ then $d y$ is the new size minus the old size, or $3(x+d x)-3 x$.
$d y=3 d x$.
So, dy is in fact three times as big as dx .
That's looking at 3 x as $\mathrm{x}+\mathrm{x}+\mathrm{x}$. We are just applying the rules for addition. $\mathrm{y}=\mathrm{x}+\mathrm{x}+\mathrm{x}$, so $d y=d x+d x+d x$.

Of course you can also think of $3 x$ as three times $x$, which means that it is a product. Products can be represented by rectangles. The area of the rectangle below is $3 x$. If $x$ grows by an infinitely small amount dx , the rectangle will grow by an infinitely thin strip along one side. That strip is shown as a blue line in the picture below:


The area of the thin blue strip is the width times the length, or $d x$ times 3 . Notice that it is only $x$ that changes, not 3 . $\mathrm{dy}=3 \mathrm{dx}$. If a constant is involved in multiplication, it sticks around and the difference is multiplied by the constant.

How does a square grow? What do you think?



The infinitely tiny change in the area is dA . The square grows by adding two infinitely thin strips to two of its sides. The increase in area is the area of those strips. The width of each strip is dx .

The length of one side of the strip is $x$, and on the other side it is $x+d x$ because one of the ends of the strip is slanted. Since we claim that $x+d x=x$, we can simply use $x$ as the length. The area of each strip is $x d x$, which means that $d A=2 x d x$.

This is our first example of an actual "changing" change. The area of a square does not change at a steady rate when the sides change. If the sides are say, 1 inch, then the area is length times width or $1 \times 1=1$ square inch. Increase the sides to 2 inches, and the area becomes $2 \times 2$ $=4$ square inches. 3 inch sides give an area of 9 square inches, and by the time we are at 4 inches for the sides the area is already at 16 square inches. The area seems to increase faster and faster when the sides grow at a steady rate. In science we see many examples where one quantity depends on the square of another quantity. With our new formula $d A=2 x d x$, we can get a very precise handle on this change. You can see that $d A$, the infinitely tiny change in the area, depends not only on $d x$, but also on the size of $x$, which is the length of the sides. When $x$ is bigger, an infinitely tiny change in $x$ produces a larger infinitely tiny change in the area.

Now let's see how a rectangle grows when both of the sides are changing. That is slightly more complex because the length is different from the width. The area of the rectangle, A , is equal to xy :


Let x increase by an infinitely small amount dx , and y by an infinitely small amount dy :


The area of the red strip is $x$ times $d y$, and the area of the blue strip is $y$ times $d x$.
$d A=d(x y)=x d y+y d x$
Again, $d(x y)$ is a single quantity even though a letter placed next to parentheses often indicates multiplication. Read it as "the difference of $x$ times $y$." This tells you how to find the difference of two changing quantities if they are multiplied. It is the original Product Rule. Notice that while the difference of $x+y$ is just $d x+d y$, the difference of $x$ times $y$ is not equal to $d x$ times dy!

Although $x$ and $y$ were used for the product rule originally, they are busy variables because they are needed for functions. Today people often use $u$ and $v$ for the product rule. Here is another rectangle, with sides $u$ and $v$ :

u
$d(u v)=u d v+v d u$
If you look carefully at this picture you can see that du is drawn larger than dv, even though both are infinitely small. If we think of the rectangle as increasing in size over time, u may be increasing faster than $v$, or the other way around.
$u$ and $v$ are meant to be changing quantities, but the Product Rule also works with constants. Let's try to find the difference of $3 x$ by using the Product Rule. Here $u$ will be 3 , and $v$ will be $x$. Because 3 is a constant, du will be zero.
$d(u v)=u d v+v d u$
$d(3 x)=3 d x+x \cdot 0$
[If you didn't learn algebra or took it long ago, please note that the $\cdot$ sign means multiplication, so $\mathrm{x} \cdot 0=0$ ]
$d(3 x)=3 d x$
We don't normally use the Product Rule when a constant is involved, because we already know that the constant will just stick around.

## Check Your Skills

Answers to previous Check Your Skills: 1. dx 2. $\mathrm{dx}+\mathrm{dw}$ 3. $\mathrm{dx}+\mathrm{dx}$, or 2 dx

1. If $y=5 x$, then $d y=$ ?
2. If $y=5 x+3$, then $d y=$ ?
3. If $y=u v$, then $d y=$ ?

## Exponents

To understand how to work with exponents in calculus we'll use a series of examples. We want to find differences when x is raised to some power, and we have already seen the first one.

## Example 1

Suppose $y=x^{2}$. What is dy in terms of $d x$ ?
Because $y$ is equal to $x^{2}$, you could think of it as the area of a square. Earlier we saw that a square grows by adding two infinitely thin strips, each with area $x d x . d y=2 x d x$.

Since $y=x \cdot x$ we can also use the Product Rule. $u=x$ and $v=x$. Both du and dv are equal to $d x$, because $d x$ is the difference of $x$ :
$d(u v)=u d v+v d u$
$d(x x)=x d x+x d x$
$d y=2 x d x$

## Example 2

$y=x^{3}$. Find $d y$.
Hmmm, this looks more complicated. $\mathrm{y}=\mathrm{x} \cdot \mathrm{x} \cdot \mathrm{x}$. How can we manage that when there are three things being multiplied instead of two? Actually, $x^{3}$ is really the volume of a cube with sides $x$. Try to imagine what would happen to the volume of that cube if $x$ increases by an infinitely tiny amount dx .


As $x$ increases by just an infinitely little bit, the width, length and height of the cube increase infinitely slightly. The cube grows by adding infinitely thin layers on three of its sides:


Let's look at one of these layers. The thickness of each layer is $d x$. The top surface measures $x+d x$ by $x+d x$, while the bottom surface is $x$ long by $x$ wide. Over the infinitely tiny distance $d x$, the surface area gets just an infinitely tiny amount larger. Because we say that $x+d x=x$, we can calculate the volume of each layer as $x^{2} \cdot d x$. There are three of these layers, so the total increase in volume is $3 x^{2} d x$.

There is also a way to get the answer that $d y=3 x^{2} d x$ without having to draw a picture. We can split $y=x^{3}$ up like this: $y=x^{2} \cdot x$

Now just multiply like we did with $u$ and $v$ :
$d(u v)=u d v+v d u$
Use $u=x^{2}$ and $v=x$ :
$d y=x^{2} d x+x \cdot d\left(x^{2}\right)$
$d y=x^{2} d x+x \cdot 2 x d x \quad$ (using our earlier result that the difference of $x^{2}$ is $2 x d x$ )
$d y=x^{2} d x+2 x^{2} d x$
$d y=3 x^{2} d x$

## Example 3

The volume of a 4-dimensional "cube" (which is called a tesseract) is represented by $y=x^{4}$.
Find dy .
Since $y=x^{3} \cdot x$, use the Product Rule with $u=x^{3}$ and $v=x$. According to the previous example, $d u=3 x^{2} d x . d v$ is still $d x$ :
$d(u v)=u d v+v d u$
$d y=x^{3} d x+x \cdot 3 x^{2} d x$
$d y=x^{3} d x+3 x^{3} d x$
$d y=4 x^{3} d x$
A regular cube grows by adding infinitely thin squares to three of its sides, and this seems to be growing by adding four cubes. That doesn't tell us what a 4-D cube looks like, but at least you can see how it grows - sort of.... ©.

This part of the story goes on and on as the powers of $x$ get higher and higher, and it always works out in the same way. Each time we take the difference, the exponent decreases by one. Does that mean we lose one dimension each time? Actually, the missing exponent is replaced by dx , so we should be able to recover it when we put things back together. We'll worry about that part later. Anyway, if you raise $x$ to the power of "any number $n$ ", the difference of $x^{n}$ will be $n x^{n-1} d x$. This is the original Power Rule.

## Check Your Skills

Answers to previous Check Your Skills: 1. 5 dx 2. 5 dx 3. udv + vdu

1. If $y=x^{2}$, then $d y=$ ?
2. If $y=3 x^{2}$, then $d y=$ ?
3. If $y=x^{5}$, then $d y=$ ?

## Division, and Cake

Calculus would not be complete if we couldn't use it with division. What if a changing quantity is divided by another changing quantity, and then we need to know the difference? If you are preparing for a calculus course, please make sure you're comfortable with the details in this section. If you are not preparing for a course just make sure you're comfortable and focus on the cake

Let's start with something simple, like 1 divided by a changing quantity x . You may or may not know that $\frac{1}{x}$ can also be written using a negative exponent: $\frac{1}{x}=x^{-1}$. Personally, I have never needed negative exponents to balance my bank account, pay my bills, or do my laundry, but they seem to be very important in calculus. Negative exponents happen naturally as you work
with positive exponents. For example, let's divide $3^{5}$ by $3^{2}$. We could change that to $\frac{243}{9}$, but it is much easier to write it like this:
$\frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3}$
If you know how to multiply fractions you can split that up into
$\frac{3}{3} \times \frac{3}{3} \times \frac{3 \times 3 \times 3}{1}$
or
$1 \times 1 \times \frac{3 \times 3 \times 3}{1}$
Normally people don't bother with this and just cross off two 3's above and below the line:

$$
\frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3}
$$

There are three 3's left on top for answer of 27. Do a few problems like this and you'll soon see a pattern. To divide numbers with exponents you take the first exponent and subtract the second one. In this example we would say that $3^{5} \div 3^{2}=3^{3}$ because $5-2=3$. That pattern inevitably leads to negative exponents for something like this: $\frac{6^{2}}{6^{5}}$, or $\frac{6 \times 6}{6 \times 6 \times 6 \times 6 \times 6}$. When you subtract the exponents you get a negative number: $6^{2} \div 6^{5}=6^{-3}$. That means that there were three 6 's left on the bottom: $\frac{1}{6 \times 6 \times 6}$. Notice that there is still a 1 on top, otherwise you'd be left with nothing divided by something, which is always zero. The 1 doesn't appear by magic; it was really already there:
$\frac{6 \times 6 \times 1}{6 \times 6 \times 6 \times 6 \times 6}$
So, $6^{-3}$ means $\frac{1}{6^{3}}$. Check out the Appendix for more info on stuff like this.

Earlier we saw that the Power Rule can be used for exponents. For $x$ raised to some power $n$, the difference is $n x^{n-1} \mathrm{dx}$. Important discoveries in mathematics have often been made by simply following patterns, and here we may wonder if that rule would work for negative exponents too. If we write $1 / x$ as $x^{-1}$ and use the Power Rule, we find that the difference is $-1 \cdot x^{-1-1} d x$, or $-1 x^{-2} d x$. And $-x^{-2}$ means $-\frac{1}{x^{2}}$, so the difference of $\frac{1}{x}$ should be $-\frac{1}{x^{2}} d x$ But is that really true?

To find an answer to that question we need to understand how a fraction changes when the bottom part changes. Suppose that we have $\frac{1}{4}$ to start, and then the bottom number increases to 5 . To find the difference between $\frac{1}{4}$ and $\frac{1}{5}$ we should subtract the old value from the new value: $\frac{1}{5}-\frac{1}{4}$. If you don't quite remember how to do that, don't worry, we have cake! The birthday cake below has been cut into 4 equal parts along its width, and each piece is $\frac{1}{4}$ :


To create fifths, I'm going to cut it into 5 equal parts along its length:


You can see that one-fifth of the cake (vertical slices) is smaller than one-fourth of the cake (horizontal slices). The cake has now been divided into four times five, or 20 pieces. One-fifth of the cake is $\frac{4}{20}$, and one-fourth of the cake is $\frac{5}{20}$. Cutting up the cake has provided a common denominator, so we can subtract the fractions. You may vaguely remember this trick as multiplying the top and bottom of each fraction by something convenient:
$\frac{1}{5} \times \frac{4}{4}=\frac{4}{20} \quad$ and $\quad \frac{1}{4} \times \frac{5}{5}=\frac{5}{20}$


The new value minus the old value is $\frac{4}{20}-\frac{5}{20}=-\frac{1}{20}$. The difference is negative because one fifth of the cake is smaller than one fourth. Increasing the denominator caused the fraction to become smaller.

Since I am naturally skeptical, and I like cake, I need to know if this also works when the denominator increases by 2 , like maybe from $\frac{1}{3}$ to $\frac{1}{5}$. Will the difference be two pieces of cake? This time I made sure to have the cake in the right proportion so we can have nice square pieces:


Yes, that works. When you compare $1 / 3$ to $1 / 5$, you end up cutting the cake into three times five, or 15 pieces (the common denominator is 15 ). One fifth is smaller than one third by two of those pieces.
$\frac{1}{5}-\frac{1}{3}$
$\frac{1(3)}{5(3)}-\frac{1(5)}{3(5)}$

$$
\frac{3}{15}-\frac{5}{15}=-\frac{2}{15}
$$

The minus sign can be placed in front of the 2 , in front of 15 , or just in front of the fraction.

To find out what happens if the denominator is $x$ and it increases by dx , I contacted Infinity Bakeries. They agreed to produce a special cake for this story, in return for some advertising. The cake is marble with chocolate frosting, and it is almost square. One of the sides is just a little bit longer. In fact, it is longer by an infinitely small amount $d x$. You could cut this cake into square pieces to get the result we are looking for, but the baker warned me not to actually take a knife to it because the special infinitely small effect is very delicate, or dangerous or something. Instead the decorator used icing to show sample cuts:


Here the total number of pieces would be $x$ times $x+d x$, so the common denominator will be $x$ times $(x+d x)$. The difference in size between $\frac{1}{x}$ and $\frac{1}{x+d x}$ is an infinitely tiny sliver. The difference is negative because $\frac{1}{x+d x}$ is smaller than $\frac{1}{x}$.

That difference is the new size minus the old, or
$\frac{1}{x+d x}-\frac{1}{x}=-\frac{d x}{x(x+d x)}$

You can see this by carefully following the pattern of the cakes, but some people prefer using algebra. If you want those details, you would multiply the top and bottom of the first fraction by $x$, and the second fraction by $x+d x$ to get the common denominator, like this:
$\frac{1}{x+d x} \cdot \frac{x}{x}=\frac{x}{x(x+d x)}$ and $\frac{1}{x} \cdot \frac{x+d x}{x+d x}=\frac{x+d x}{x(x+d x)}$
Now we can subtract:
$\frac{x}{x(x+d x)}-\frac{x+d x}{x(x+d x)}$
Be careful to subtract both $x$ and $d x$ (always use parentheses). The result of that is negative: $x-x-d x=-d x$. So our conclusion is that
$\frac{1}{x+d x}-\frac{1}{x}=\frac{-d x}{x(x+d x)}$
Algebra sensibly agrees with the cake, so we get the same answer. However, it seems that the infinitely small sliver $d x$ has made $x$ infinitely large by comparison. We claim that $x+d x=x$, which in this case would mean that both sides of the cake are actually the same length. The common denominator turns into $x^{2}$ :
$\frac{1}{x+d x}-\frac{1}{x}=-\frac{d x}{x^{2}}$
Because $-\frac{d x}{x^{2}}$ can also be written as $-\frac{1}{x^{2}} \cdot \frac{d x}{1}$, or $-\frac{1}{x^{2}} d x$, our earlier use of the Power Rule did in fact produce the correct answer.

If $y=\frac{1}{x}$, then $d y=-\frac{1}{x^{2}} d x$
It is easy to get distracted by a bunch of calculations, either because they're too hard and you don't get it, or because you feel they're so easy that you totally get it. But the real focus here is cake, and specifically the cake pictured. Is it actually square or isn't it? How did they even make it? And how can you just look at a cake with chocolate frosting and not eat it? All of these are important questions, especially that last one. We weren't supposed to cut the cake, but maybe breaking just a little piece off the corner wouldn't hurt....

Mmmm, chocolaty, with just a hint of ...infinity?? Oh no, now I feel really, really full. That tiny piece must have been a lot bigger than it looked!

## Check Your Skills

Answers to previous Check Your Skills: 1. $2 x d x \quad$ 2. $6 x d x \quad 3.5 x^{4} d x$

1. If $y=\frac{1}{x}$, then $d y=$ ?
2. If $y=\frac{3}{x}$, then $d y=$ ?
3. If $y=\frac{1}{7}$, then $\mathrm{d} y=$ ?

## The Quotient Rule

Remember that we were looking for a way to use calculus with division. If a changing quantity u is divided by a changing quantity v , we can write that as $\mathrm{u} / \mathrm{v}$, but also as a multiplication: u times $1 / v$. Now that we know that the difference of $1 / v$ is $-\mathrm{dv} / \mathrm{v}^{2}$ we have enough information to just apply the rule for multiplication.

On the other hand, a more direct approach shows how calculus really works. Again, it is the principle that matters rather than the detailed calculations. Just like we said before, the difference is the new value minus the old value. Let $u$ increase by an infinitely tiny amount du, while $v$ increases by an infinitely tiny amount $d v$. Then subtract the original value $u / v$ :
$\frac{u+d u}{v+d v}-\frac{u}{v} \quad$ (the "new" value minus the "old" value)
Create a common denominator by multiplying the top and bottom of the first fraction by $v$, and the top and bottom of the second fraction by $(v+d v)$.
$\frac{u+d u}{v+d v} \cdot \frac{v}{v}$ and $\frac{u}{v} \cdot \frac{v+d v}{v+d v}$
Now it looks like this:
$\frac{v(u+d u)}{v(v+d v)}-\frac{u(v+d v)}{v(v+d v)}$
But to subtract we have to multiply out the stuff on top. Just like $3(10+2)$ is $30+6$, $v(u+d u)$ is the same as $v \cdot u+v \cdot d u$. We get
$\frac{u v+v d u-u v-u d v}{v(v+d v)}$
Be careful to subtract both uv and udv. That leaves
$\frac{v d u-u d v}{v(v+d v)}$
Now that the calculations are done we claim that $\mathrm{v}+\mathrm{dv}=\mathrm{v}$ :
$d\left(\frac{u}{v}\right)=\frac{v d u-u d v}{v^{2}} . \quad$ This is the original Quotient Rule.

While the original Product Rule can easily be seen as a growing rectangle, this formula only looks like an abstract collection of letters. I was just going to move on to the next section, but then I remembered something that happened a long time ago. I was visiting an indoor garden when I came across an intricate metal sculpture. It looked like an abstract collection of spikes with chunks of metal attached to them. Careful examination suggested some purpose to it, but I really couldn't see what it was. "I just don't get any of this modern art," I thought, but fortunately there was someone around who saw the world a little differently. Kamex was only six years old at the time, wandering aimlessly among the plants. I gently guided him to the sculpture. "What do you think of this?" I asked, not sure if there would be an answer at all. Surprisingly, he froze. His eyes didn't move as he stared straight ahead, and it wasn't clear if he was even noticing the object in front of him.

As the minutes ticked by, I started to worry. Surely no young child could remain motionless that long. Was this some type of seizure? Had I done something wrong? Should I move him away from the sculpture? Suddenly his face broke into a smile of understanding. "They're still working on it," he announced happily. "Well, it could use some work all right," I thought, annoyed with the artwork for wasting my time. As I turned to leave, I noticed a tiny sign near the bottom of the sculpture. "Genesis," it said. Genesis ... as in the first book of the Bible ..., creation ..., they're still working on it... . Suddenly the pieces of metal seemed to move along their spikes, coming together to make a sphere with continents and oceans, creating the Earth. Now why hadn't I seen that before?

Unlike Kamex I can't compose music, write novels, or use my detailed knowledge of random topics to fix someone's unusual speech problem. But what I can do is learn to look at something without assuming that it's just too hard to understand.

When I look at the original Quotient Rule again, I can see that the square on the bottom kind of isn't perfectly square:
$\frac{v d u-u d v}{v^{2}}=\frac{v d u-u d v}{v(v+d v)}$
The formula also says that it is using infinitely small differences, but it was just made by subtracting fractions. Those differences are not special in any way, so the whole thing should work with regular stuff like maybe an apple pie. The difference between half a pie and three quarters is one quarter, and you can actually use the Quotient Rule formula to see that. To avoid negative values, we will imagine that $1 / 2$ increases to $3 / 4$. (Although sadly apple pies always seem to decrease in size over time, the formula does actually work either way). $\frac{\mathrm{u}}{\mathrm{v}}$ will be $\frac{1}{2}$.

The original Quotient Rule:
$\frac{v d u-u d v}{v(v+d v)}$
Replace infinitely small differences with regular differences (indicated by the Greek letter $\Delta$ ):
$\frac{v \Delta u-u d \Delta v}{v(v+\Delta v)}$
As $\frac{1}{2}$ changes to $\frac{3}{4}$, both the top and the bottom increase by 2 . Fill in the values:
$\frac{2 \cdot 2-(1 \cdot 2)}{2(2+2)}$
Simplify:
$\frac{4-2}{2(2+2)}=\frac{2}{2(2)}=\frac{2}{8}$
And $2 / 8$ is the same as $1 / 4$. That complex-looking formula is actually just another way to subtract fractions, and you can check that it works with any two fractions. The abstract, and amazing, part of the Quotient Rule formula is that it works even when the difference between the two fractions is infinitely small!

This completes how addition, subtraction, multiplication, exponents and division work in calculus.

## Check Your Skills

Answers to previous Check Your Skills: 1. $-\frac{d x}{x^{2}}$ or $-\frac{1}{x^{2}} d x \quad$ 2. The difference is 3 times as big as the difference of $\frac{1}{x}$. Write that as $3 \cdot-\frac{d x}{x^{2}}$ or $-\frac{3 d x}{x^{2}}$ or $-\frac{3}{x^{2}} d x \quad 3.0$, because $\frac{1}{7}$ is a constant.

1. True or False: If $z=x+y$, then $d z=d x+d y$
2. True or False: If $z=x \cdot y$, then $d z=d x \cdot d y$
3. True or False: If $z=x / y$, then $d z=\frac{y d x-x d y}{y^{2}}$

## Rates of Change

Consider the following problem:
One morning, Mike is late for school and the tires of his car squeal a bit as he pulls away from the curb. His vehicle is accelerating at 3 meters/second per second. Ten meters ahead of him on the sidewalk is a snail moving in the same direction at 0.2 millimeters/second. The snail is tired after a long night of whatever it is that snails do, and it is decelerating at $0.01 \mathrm{~mm} / \mathrm{sec}$ per second as it approaches a suitable place to sleep. What is the speed of the car when it overtakes the snail?

If you are going to take a calculus course you will get to solve irrelevant problems like this, because for obvious reasons they won't let you near anything more critical like finding the tension on the wires of a suspension bridge or the load on an airplane wing at take-off.

So, let's take a closer look at this situation. Both the car and the snail are changing position, but the car is doing so much faster. Would that mean that an infinitely small difference in the position of the car, dc , is much larger than an infinitely small difference in the position of the snail, ds? The answer to that is yes if you look at it relative to time. And of course we do usually think of such change as happening over time. In this case we would need to find the speed of the car and the snail, but those speeds are changing constantly. To get around this problem, we will divide time itself into infinitely many infinitely small portions. Textbooks normally use the letter $t$ to represent time, so an infinitely small change in time is dt . During this infinitely tiny increment of time, the car changes its position by an amount dc. In that same infinitely tiny bit of time, the snail also moves, but of course not as far as the car. dc is infinitely small, but ds is smaller.

Anyway, as we said earlier, speed is distance (the change in position) divided by time. The speed of the car at any given point in time is $\frac{\mathrm{dc}}{\mathrm{dt}}$. Even though you are dividing something infinitely small by something else infinitely small, you will actually get a real value for the speed. Now it no longer matters that the speed is changing all the time, because each interval of time dt is so small that any change that occurs during the interval is negligible. It also wouldn't matter if the speed is not changing at all, because $\frac{\mathrm{dc}}{\mathrm{dt}}$ still represents the speed even when the car is no longer accelerating. In the same way, $\frac{\mathrm{ds}}{\mathrm{dt}}$ will give an exact value for the speed of the snail at any given moment. Ratios like $\frac{\mathrm{dc}}{\mathrm{dt}}$ and $\frac{\mathrm{ds}}{\mathrm{dt}}$ are called derivatives, and they tell us how fast something is changing. Derivatives measure the rate of change. Because we are
considering an infinitely short period of time, we will get the rate of change at a particular moment. This is called the instantaneous rate of change.

Mathematicians and scientists normally use derivatives rather than differences, because they already know that things are changing; they just need to measure the change and do calculations with it.

Looking back at multiplication and the...what was that? Oh, you were expecting an answer to that problem with the car and the snail? Look again, the instructions direct you to consider the problem, and we considered it. Sometimes we get so caught up in finding the answer that we stop thinking about what we are really doing. For one thing, that snail is moving insignificantly slowly compared to the car. Also, the problem was made to show rates of change, but those rates themselves are actually changing at a constant rate. Acceleration (or deceleration), is defined as the change in speed over time. If the acceleration is constant, the speed is changing in a constant and easily predictable way, for both the car and the snail. Physics is very good at handling this with regular math. For more details, see your local physics teacher. It is not unusual to find calculus problems like this that really don't need calculus at all, but that seems to be a secret so don't tell anyone. Sshhh....

Now where were we? Oh yes, multiplication and the original product rule.


As we noted earlier, du in this picture is drawn larger than $d v$. If $u$ and $v$ are both changing over time, one may be changing faster than the other. So, the rate of change of $u$, represented by $\frac{\mathrm{du}}{\mathrm{dt}}$, is larger than the rate of change of $v, \frac{\mathrm{dv}}{\mathrm{dt}}$. It is possible that both $u$ and $v$ are changing at a constant rate, but they could also be changing in a much more complex way. In that case it would be difficult or impossible to use regular math to determine how fast the area of the
rectangle is growing or shrinking. Don't worry, calculus can handle it. In fact, you should probably never worry about how the area of a rectangle is changing because that could be bad for your mental health. We already figured out that the change in the area, $d A$, can be calculated like this:
$d(u v)=u d v+v d u$
If we want to know how fast the area is changing we need the rate of change (the derivative), $\frac{\mathrm{dA}}{\mathrm{dt}}$. Because we are using an infinitely tiny interval of time to calculate that change, it won't matter whether the area is changing at a constant rate or not. $d A$ is equal to $d(u v)$, and we already know that $d(u v)=u d v+v d u$. Since regular math still works with calculus we can get $\frac{d A}{d t}$ by dividing both sides by dt :
$d A=d(u v)$
$\frac{\mathrm{dA}}{\mathrm{dt}}=\frac{\mathrm{d}(\mathrm{uv})}{\mathrm{dt}}=\frac{\mathrm{udv}+\mathrm{vdu}}{\mathrm{dt}}$

If you remember your fractions, you can rewrite that last part.
Just like $\frac{1+3}{5}$ or $\frac{4}{5}$ is the same as $\frac{1}{5}+\frac{3}{5}, \frac{u d v+v d u}{d t}$ can be split up into $\frac{u d v}{d t}+\frac{v d u}{d t}$.
And $\frac{\mathrm{udv}}{\mathrm{dt}}$ is really the same thing as $\mathrm{u} \cdot \frac{\mathrm{dv}}{\mathrm{dt}}$, so:
$\frac{d A}{d t}=u \frac{d v}{d t}+v \frac{d u}{d t}$

This rearrangement shows how the rate of change in the area of the rectangle is related to how fast $u$ is changing, $\frac{d u}{d t}$, and also to the rate of change in $v, \frac{d v}{d t}$. This more modern version of the Product Rule may look complicated, but after the first 100 or so problems you do with it, it will seem a bit easier.

Physicists and ordinary people normally consider change as happening over time, but mathematicians prefer to express changes in terms of $y$ and $x$. Maybe that is because they like
to graph things in an $x-y$ coordinate system, or maybe they don't consider time a reliable measure of change as it can practically stand still inside a math classroom.

Anyway, it isn't a problem because we have already looked at changes in y and x . Let's go back to one of our first examples, $y=3 x$. Using lines and a rectangle, we figured out that dy has to be three times as big as $\mathrm{dx}: \mathrm{dy}=3 \mathrm{dx}$. To get a rate of change, we have to ask, "What is the change in y per unit of change in $x$ ?" The best way to see that is by using a graph.


At every point on this graph the value of $y$ is three times the value of $x$, but in calculus we are thinking in terms of change instead of just static points. Although one of the advantages of using $x$ rather than time is that we can think of it as either growing or shrinking, people usually imagine $x$ as increasing. Start at the left side of the graph and move right to imagine the change in $y$, so that you can see that $y$ is constantly increasing. The rate of change is positive. Because this is a nice straight line, we can measure the rate of increase by taking any two points and dividing the change in $y, \Delta y$, by the change in $x, \Delta x$. This is also the slope of the line, "rise" over "run", which is 3 (three units of $y$ per unit of $x$ ).

Rate of change $($ slope $)=\frac{\Delta y}{\Delta x}=3$
The slope is the same everywhere, so if we select two points on the line that are infinitely close together, the change in $y$ and the change in $x$ will be infinitely small and we can use $d y$ and $d x$. We already know that $d y=3 d x$. Dividing both sides by $d x$, we get $\frac{d y}{d x}=3 \frac{d x}{d x}$. Since $\frac{d x}{d x}$ is just 1 , the slope is still 3 :

Rate of change $=\frac{d y}{d x}=3$
$\frac{d y}{d x}$ is a derivative, and it tells us the rate of change of $y$. $y$ changes at a steady rate of 3 units for every unit of change in $x$. That isn't a very spectacular conclusion because you can just see it from the graph, but it becomes important when a graph is more complex.

Here is a graph of $y=x^{2}$ :


Again, you'll be expected to look at this picture dynamically in terms of a change in $x$. Start at the left side. As $x$ increases (it becomes less negative), $y$ is decreasing. The rate of change of $y$ relative to x is negative. y decreases rapidly at first, and then more slowly, until x reaches zero.

Now $y$ starts to increase as $x$ continues to increase. The rate of change is positive. $y$ increases slowly at first and then faster and faster.

The rate of change would be the slope, or the change in $y$ divided by the change in $x$, but to determine that we would need a straight line. Fortunately, as we said earlier, if you zoom in on a curve infinitely far it becomes just like a straight line. Looking at an infinitely small piece of this curved graph we can determine the rate of change by dividing $d y$ by $d x$. We already know that if $y=x^{2}$, then $d y=2 x d x$. Now just divide both sides by $d x . d x$ cancels out on the right side of the equation so you get $\frac{d y}{d x}=2 x$. The derivative of $y=x^{2}$ is $2 x$, which means that when $x$ is negative the rate of change is negative, and when $x$ is positive and increasing $y$ gets larger faster and faster. Even though the rate of change isn't steady, thanks to calculus we can still measure it precisely at any given point $x$.

Derivatives work just like differences. Because differences can just be added or subtracted, so can derivatives. For example, if $y=x^{3}+2 x^{2}-6 x$ then the derivative $d y / d x$ is $3 x^{2}+4 x-6$. By the way, since the difference of a constant is zero, its derivative is zero too. If something never changes, then its rate of change is zero. If $y=5$, then $d y$ is zero, and $d y / d x$ is still 0 .

## Check Your Skills

Answers to previous Check Your Skills: 1. True 2. False, $d z=x d y+y d x$ 3. True

1. If $y=x^{2}$, then $\frac{d y}{d x}=$ ?
2. If $y=3 x^{2}$, then $\frac{d y}{d x}=$ ?
3. If $v=2 t$, then $\frac{d v}{d t}=$ ?

## Symmetry

Sometimes change is better expressed relative to something other than $t$ or $x$. For example, the change in the area of a circle is best expressed in terms of a change in its radius $r$. The area of a circle is given by the formula $A=\pi r^{2}$. You may have seen this formula before, but you may not know that it can be constructed by dividing a circle into infinitely many infinitely small parts. Let's take a look at that.

In ancient times a circle was measured by taking a string or rope to find the longest distance across it, which is the diameter. Because a circle is such a fundamental shape, people expected that the ratio between the diameter and the circumference should be either a nice whole number or some special fraction. But when they tried to measure the circumference with a string that was the length of the diameter they found that three of those lengths come up a bit short. In fact, the circumference of a circle is about 3.14 times the diameter. Many attempts to measure or calculate this ratio failed to find a precise value, even though it has to be a number that actually exists! That number came to be known as pi $(\pi)$, and it is now known that we can neither write it as a decimal number nor express it as a fraction. Pi is a very special number, and the circumference of a circle is $\pi$ times the diameter $D$ :

## Circumference of a Circle $=\pi D$

That's an easy thing to remember if you think of how it was discovered. These days we tend to use the radius $r$ rather than the diameter to measure and describe a circle. Since the radius is only half of the diameter, the circumference of a circle is $2 \pi r$.

To find the area of a circle with radius $r$, imagine that you cut it up into 8 slices, like you would cut a pizza. Then cut each slice in half so you have 16 thinner slices. Lay the slices in a row, alternating the top and bottom, like this:


Although this is not a rectangle, and the edges of the slices are curved, you could still make some estimate of the area. The width is of this shape is approximately $r$, and the length is about half the circumference of a circle, or about $\pi r$. Now imagine that instead of 16 slices, we cut the circle into infinitely many slices. These slices no longer seem to have a curved edge. In fact, they look kind of like very thin lines with a straight top edge. Putting these slices into the same pattern (assuming you have an infinite amount of time available), you would find that your new shape is a rectangle with width $r$ and length $\pi r$. The area is $\pi r \cdot r$ :

Area of a Circle $=\pi r^{2}$
When the radius of a circle increases by an infinitely tiny amount dr, the area increases by an infinitely tiny amount $d A$. $d A$ is represented by the blue ring on the outside of the circle in the picture below:


Imagine that you could take the infinitely thin blue ring, cut it somewhere and then lay it out like a long strand of really thin spaghetti. One side of the strand represents the inner border of the blue ring, $2 \pi r$, and the other represents the outer border, $2 \pi(r+d r)$. Because $d r$ is infinitely small both sides actually have the same length, creating a perfect long thin rectangle. The width of that rectangle is $d r$, and the length is $2 \pi r$. The change in the area, $d A$, is the width times the length, $2 \pi r d r$. This is the same result that you would get by using the Power Rule. If $A=\pi r^{2}$, then $d A=\pi$ times $2 r d r$, since $\pi$ is a constant. Yes, $\pi$ is a constant even though it is a strange kind of number, since it is always the same.

As the radius changes, the rate of change of the area of the circle is $\frac{\mathrm{dA}}{\mathrm{dr}}$. To get this rate, we divide both sides of $d A=2 \pi r d r$ by $d r$ :
$\frac{\mathrm{dA}}{\mathrm{dr}}=2 \pi r$
The rate of change of the area of a circle (per unit change in the radius) is equal to the circumference.

Notice that the actual rate depends on the radius. When the circle is small, the area increases more slowly than when the radius is larger.

Now, I have a question. Why did that circle grow all the way around, while the square we looked at before only grew on two sides? When you have questions like that, it is fine to just play around with your math to find answers. After all, math belongs to everyone, and it's not like you could damage it by experimenting. The square grew on two sides because each side $x$ can only grow on one end. What if we made the square grow in a different way? Let's consider the short "radius" of the square, which is officially called the apothem:


Here $r$ extends from the center of the square, so the length of $r$ is one-half of the length of the side. The area of the square is now $(2 r)^{2}$, which is $2 r$ times $2 r$, or $4 r^{2}$. When $r$ increases, the area increases too. If $A=4 r^{2}$, then $d A=8 r d r$ (according to the Power Rule). In the picture below you can see that $d A$, the infinitely tiny increase in the area, is 4 times $2 r d r$ :


Now the square grows on all four sides (8). If $d A=8 r d r$, we can divide both sides by $d r$ to get a rate of change: $\frac{\mathrm{dA}}{\mathrm{dr}}=8 \mathrm{r}$. The rate of change of the area, for any given value of $r$, is equal to the "circumference", which in this case is the perimeter.

If you have a lot of time on your hands, or if you feel sad that there is no "Check Your Skills" for this section, you may want to do the same thing with a cube. The "radius" of the cube would be $r$, and the volume $V$ would be $(2 r)^{3}$, which is $2 r$ times $2 r$ times $2 r$. If $V=8 r^{3}$, then the Power Rule tells us that $\mathrm{dV}=8$ times $3 r^{2} \mathrm{dr}$, which is $24 \mathrm{r}^{2} \mathrm{dr}$. As before, the cube grows by adding infinitely thin layers, but now those layers are on all 6 faces. Each layer has surface of $2 r$ by $2 r$, or $4 r^{2}$, and a volume of $4 r^{2} d r$. When the cube grows infinitely slightly on all 6 of its faces, the infinitely tiny change in the volume, $d V$, is $24 r^{2} d r$. We could also say that $\frac{d V}{d r}=24 r^{2}$. The surface area of the cube is 6 faces times the surface area of each face, or 6 times $4 r^{2}$. The rate of change of the volume is equal to the surface area.

In the same way, you can make a four-dimensional cube grow on all of its...uh...cubes. Earlier, we made a 4-D cube grow by adding four cubes. Using $r$, we can express the volume as $(2 r)^{4}=16 r^{4}$. Now use the Power Rule: $d V$ is 16 times $4 r^{3} d r$, which is $64 r^{3} d r$. Each growth cube will have a volume of $8 r^{3} \mathrm{dr}$, so there must be 8 cubes that make up the "surface" of the shape that is the 4-D equivalent of a cube. I don't know about you, but I can't imagine that at all.

## Reassembly

In general, it is much easier to take something apart into very small pieces than to put it back together, as I found to my dismay when I tried to fix my toaster. Really, how was I supposed to know that those little metal wires that hold the bread would all fall out when I opened the thing? Stuff goes wrong in math too, but normally those mistakes end up crumpled in the trash. That isn't necessarily a good thing, because it makes it look like maybe you're the only one who finds it hard. In this story you will see an actual error so that you can learn from someone else's mistake, which is always preferable to learning from your own.

Previously, we took apart $y=x^{2}$ into infinitely many infinitely tiny pieces: $d y=2 x d x$. Now it is time to put things back together. As we saw before, infinitely many dy's can be summed back up to make $y$. The sum of all the dy's is $\int$ dy. Remember that $\int$ is just a fancy $S$ that stands for the sum, and that sum is called an integral. The sum of all of the infinitely tiny parts of $y$ is $y$ itself: $\int d y=y$. That should mean that $\int d y=\int 2 x d x$, and I should be able to take all those pieces that are $2 x d x$ and add them to get a square with an area of $x^{2}$. Because it seems a little complicated I am just going work with x dx first, and then make all the pieces twice as long. Here is one piece that is $x$ long and $d x$ wide:

## X

I can stack these pieces on top of each other until the width is the sum of all the dx 's, which is x . That gives me a square $x$ units long and $x$ units wide:


Then I just multiply by 2, and ... wait, now I have two squares instead of one! Nooo.....!! You may think math is boring, but you probably haven't experienced the sinking feeling you get in the pit of your stomach (or maybe the pit of your brain?) when you have taken reality apart and can't get it back together.

What went wrong?? When I was taking a calculus course and mindlessly applying formulas all of this worked just fine, so why wouldn't it stand up to thinking about it in a reasonable way? If something like this happens ever happens to you, take a deep breath and tell yourself that it is math, so it has to work. Just like I eventually managed to get the toaster back together, it finally occurred to me what my mistake was. It is easy to forget that calculus deals with changing quantities, so $x$ is not just one size. When you first start putting the pieces back together x is very small, and it increases as you go along:


The sum of all those pieces $x d x$ is one half of the square $x^{2}$. Phew, that's better. Now I can be sure that two times the sum of $x d x$ is $x^{2}$ :
$2 \int x d x=x^{2}$
On the other hand, as I realized much later, I could have saved myself all that trouble by imagining a square growing from the start:


Each time the square grows an infinitely tiny bit, the area increases by $2 x d x$. The sum of all the increments $2 x d x$ is the square $x^{2}: \int 2 x d x=x^{2}$.

At least this shows that you can put the number 2 either inside or outside of the sum sign:

## $2 \int x d x=\int 2 x d x$

That makes sense, because that constant never got divided up, and it isn't changing, so it doesn't have to be summed up in some way. Instead, the sum is simply multiplied by the constant.

We can also take $d y=3 x^{2} d x$ and sum up the pieces to get back to $y=x^{3}$. On the left side of the equation, the sum of all the $d y^{\prime} s$ is $y: \int d y=y$. On the right side, ignore the 3 for a moment and make some pieces of size $x^{2} d x$. These pieces are infinitely thin squares that increase in size as $x$ grows. Just stack them all on top of each other, with the largest one at the bottom so your stack doesn't fall over. Now you have a pyramid with a base of $x^{2}$ and a height of $x$. If you remember your geometry, you will know that the volume of a pyramid is $1 / 3$ that of the box that would contain it ( $1 / 3$ base times height). If you don't remember any geometry, you can always look that kind of stuff up online. In this case the volume of the pyramid is $1 / 3$ times the base, $x^{2}$, multiplied by the height $x$. That makes the volume $\frac{1}{3} x^{3}$. If we then multiply by 3 we'll get $x^{3}$, so
$\int 3 x^{2} d x=3 \int x^{2} d x=x^{3}$
When it comes to 4 dimensions, regular geometry doesn't give us such quick answers. However, by now you can see a pattern and you will likely guess that since the difference of $x^{4}$ is $4 x^{3} d x$, the sum $\int 4 x^{3} d x$ must be $x^{4}$. This pattern suggests something interesting. The difference of $x$ is $d x$, and the sum of the infinitely many $d x^{\prime}$ s completely fills the onedimensional space represented by $x$. The difference of $x^{2}$ is $2 x d x$, and the sum of the increments $x d x$ fills half the two-dimensional space $x^{2}$. The difference of $x^{3}$ is $3 x^{2} d x$, and $\int x^{2} d x$ fills one third of the three-dimensional space $x^{3}$. So, if we stack up a 4-dimensional "pyramid" using blocks of increasing size $x^{3} d x$, that should fill $1 / 4$ of the available 4D space. If you've always wanted to build a 4D shape, now you can. Just make sure to use the special cubes that extend just infinitely slightly into the fourth dimension

We also took apart a circle and found that $d A=2 \pi r d r$. To put the circle back together we can sum up all of the infinitely small parts of size dA that make up the area. Just take the sum on both sides of the equation. $A=\int d A$, and since $d A=2 \pi r d r, \int d A=\int 2 \pi r d r$. The numbers 2 and $\pi$ are constants, so they don't need to be summed up. $\int 2 \pi r d r=2 \pi \int r d r$. Adding up all of the parts $r$ dr gives $1 / 2 r^{2}$, just like we found that $\int x d x=1 / 2 x^{2}$. Therefore, $2 \pi \int r d r=2 \pi \cdot 1 / 2 r^{2}$
$=\pi r^{2}$. And the area of a circle is actually $\pi r^{2}$. All of those infinitely thin rings add back up to a whole circle!


There is however something that will never go back together in quite the same way. Remember poor Myra and her 4860 calculus problems? Because the number of problems was a constant that was by itself, and not being used to multiply a changing quantity, it ended up as just zero. So now how do we reassemble that? Calculus ate my homework?? The truth is that when we took the difference, some information was lost. In fact, any time we sum things back up there could have been some constant present along with any differences that we found, and it is no longer possible to determine how big that constant really was. In this case that is actually for the best. "Dear Myra, calculus has erased your memory. Your textbook contains C problems, where C is some constant. It is not a good idea for you to determine the value of C . Just do your assigned problems each week, and everything will be fine." Now Myra is no longer feeling discouraged by the incredibly large number of problems in her book.

If $y=x$, then $d y=d x$. But If $y=x+4, d y=d x+0$, so $d y$ is still equal to $d x$. To account for this we always add an unknown constant $C$ when we reassemble things. If I have an infinite number of pieces of size $d x$, I can sum them back up with an integral like this: $\int d x$. That will reassemble them back into $x$, but there could have been a constant present before things were taken apart. That means I should write $\int d x=x+C$, rather than just $\int d x=x$. Maybe there was no constant, but then C would just be zero, and the answer will still be correct.

## Check Your Skills

Answers to previous Check Your Skills:<br>1. $2 x$<br>2. $6 x$<br>3. 2

1. $\int \mathrm{dr}=$ ?
2. $\int 2 \mathrm{dx}=$ ?
3. $\int x d x=$ ?

## The End

Congratulations on finishing this brief overview of calculus. You now know some basics about finding a derivative and putting differentials back together with an integral. Calculus can show you some amazing things, like the infinitely small differences that are (or not) everywhere in your changing world, 4D shapes that may or may not exist somewhere, and infinitely large cakes that will take forever to eat. Mostly though it is a tool that is used every day by engineers, scientists, financial analysts, computer programmers, and students struggling to get through a calculus course. If you find yourself in that last group, try to stay focused on what you are actually doing, and why you can solve problems in the way that you are taught to do. The details may be difficult, or boring, or both, but the ideas of calculus are really awesome.

May you live, and learn, happily ever after. The end.

## Answers to Final Check Your Skills

Did you remember to add the constant C to your answers?

1. $r+C$ 2. $2 x+C$. This integral is twice the size of $\int d x \quad 3 . \frac{1}{2} x^{2}+C$

## Appendix: Exponent Review

If you take a calculus course, you'll be expected to know a lot about exponents. Those things are summarized in this review section.

The variable x is feeling a bit worn out after being divided up and put back together multiple times, so the numbers 9 and e (which is a number that is sort of like $\pi$ ) have kindly volunteered to help out.

Exponents are the second item in "Please Excuse My Dear Aunt Sally". This means that they have a very high priority in the order of mathematical operations. An exponent on something generally belongs exclusively to that one thing, unless there are parentheses present. So, $9 \mathrm{e}^{2}$ is just $9 \mathrm{e}^{2}$, because the exponent applies only to the e. However, once you place some parentheses they get priority, and the exponent applies to the entire thing inside those parentheses. $(9 \mathrm{e})^{2}$ means $9 \mathrm{e} \cdot 9 \mathrm{e}$ which is $81 \mathrm{e}^{2}$.

First we will do some multiplication involving exponents, which is the easiest part. $9^{3} \cdot 9^{4}=$ ? This really means $9 \cdot 9 \cdot 9$ times $9 \cdot 9 \cdot 9 \cdot 9$. Now you have seven 9 's in a row, which is $9^{7}$. For multiplication, just add the exponents.

For division, we can subtract the exponents. $\frac{9^{5}}{9^{3}}=\frac{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}{9 \cdot 9 \cdot 9}=9^{2}$. Notice that by subtracting exponents, we find that $\frac{9^{1}}{9^{1}}$ is $9^{1-1}$ which is $9^{0}$. Since any number divided by itself is $1,9^{0}$ must mean 1. That works for every other number too, except for zero. Since we can't divide by zero, $0^{0}$ is really undefined. You might think that $0^{0}$ should just be 1 too, but on the other hand 0 to any other power is 0 . When something is not defined, you can often get a handle on it by sneaking up on it: $0.25^{0.25}=0.70710 \ldots, 0.1^{0.1}=0.79432 \ldots, 0.01^{0.01}=0.95499 \ldots$. The limit of this process is 1 . If you really need to have a value for $0^{0}$, you should probably use 1 .

Another thing that happens when you start subtracting exponents is that you end up with negative exponents. Consider $\frac{9^{2}}{9^{5}}$. You can write that as $\frac{9 \cdot 9}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}$, and cross off two 9 's on the top and bottom. Be careful because there is still a 1 on the top after you are done with that: $\frac{9 \cdot 9 \cdot 1}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}=\frac{1}{9^{3}}$. If you had subtracted the exponents to begin with that would leave you with $9^{-3}$. This tells you that $9^{-3}$ means $\frac{1}{9^{3}}$.

You can also put an exponent on something that already has an exponent. This looks like $\left(9^{2}\right)^{3}$. Thinking carefully about what that means, we rewrite it as $9^{2} \cdot 9^{2} \cdot 9^{2}=9^{6}$. So, when you
raise a power to another power, you multiply the exponents. Be careful when there are multiple things inside the parentheses. $(9 \mathrm{e})^{3}$ means $9 \mathrm{e} \cdot 9 \mathrm{e} \cdot 9 \mathrm{e}$, or $9^{3} \mathrm{e}^{3}$.

What about exponents that are fractions? Let's consider $9^{\frac{1}{2}}$. We know that when we multiply things with exponents, we can just add the exponents. So $9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}=9^{\frac{1}{2}+\frac{1}{2}}=9^{1}$. This means that $9^{\frac{1}{2}}$ has to represent a number that, when multiplied by itself, is 9 . The only candidate for this is $\sqrt{9}$, since $\sqrt{9} \cdot \sqrt{9}=9$. So what is $9^{\frac{1}{3}}$ ? There would have to be a number such that $9^{\frac{1}{3}} \cdot 9^{\frac{1}{3}} \cdot 9^{\frac{1}{3}}=9$. This is the number we call the cube root of 9 , or $\sqrt[3]{9}$. By now you can probably guess that $9^{\frac{1}{4}}$ is $\sqrt[4]{9}$, and so on.

Calculus also involves more complicated fractional exponents like $9^{\frac{3}{2}}$. Just use your knowledge of fractions to see how such an exponent could have been created. We know that when we raise a power to a power, the two numbers are multiplied. That means that $9^{\frac{3}{2}}$ could have been created in two ways: $\left(9^{3}\right)^{\frac{1}{2}}$ or $\left(9^{\frac{1}{2}}\right)^{3}$. Therefore, $9^{\frac{3}{2}}=\sqrt{\left(9^{3}\right)}=(\sqrt{9})^{3}$. Both these forms mean the same thing. Notice that it is the denominator of the fractional exponent that determines what kind of root is involved. Scary-looking fractional exponents follow the same rules as regular exponents. Add them when you multiply, and subtract them when you divide.

## Summary

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\begin{array}{ll}
9^{a} \cdot 9^{b}=9^{a+b} & \left(9^{a}\right)^{b}=9^{a b} \\
\frac{9^{a}}{9^{b}}=9^{a-b} & (9 e)^{a}=9^{a} e^{a} \\
9^{-a} \text { is } \frac{1}{9^{a}} & 9^{\frac{1}{n}}=\sqrt[n]{9} \\
9^{0}=1 & 9^{\frac{a}{n}}=\sqrt[n]{9^{a}} \text { or }(\sqrt[n]{9})^{a}
\end{array}
$$

